Matrix Notation for the Regression Notes of DB DeLury

References:
SR Searle (1982) : Matrix Algebra
K Hoffman & R Kunze (1961): Linear Algebra
DAS Fraser (1976) : Probability and Statistics: Theory and Applications

Notation p1

$y \quad x_0 \quad x_1 \quad x_2 \; . . . \; x_p \qquad \mathbf{x}' = (x_0 \quad x_1 \quad x_2 \quad . . . \quad x_p)$

$y_\alpha \quad x_{0\alpha} \quad x_{1\alpha} \quad x_{2\alpha} \; . . . \; x_{p\alpha} \qquad \mathbf{x_\alpha}' = (x_{0\alpha} \quad x_{1\alpha} \quad x_{2\alpha} \quad . . . \quad x_{p\alpha}) \quad \alpha = 1 ... N$

This data can be organized as a column vector $\mathbf{y}$ and a matrix $\mathbf{X}$ with N rows and p+1 columns. At times, there is advantage to viewing $\mathbf{X}$ as a column of N row vectors $\mathbf{x_\alpha}'$ each of length p+1 or as a row of p+1 column vectors $\mathbf{x_i}$ each of length N.

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ . \\ . \\ . \\ y_N \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} x_{01} & x_{11} & . . . & x_{p1} \\ x_{02} & x_{12} & . . . & x_{p2} \\ x_{03} & x_{13} & . . . & x_{p3} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ x_{0N} & x_{1N} & . . . & x_{pN} \end{pmatrix} = \begin{pmatrix} \mathbf{x_1}' \\ \mathbf{x_2}' \\ \mathbf{x_3}' \\ . \\ . \\ . \\ \mathbf{x_N}' \end{pmatrix} = (\mathbf{x_0} \quad \mathbf{x_1} \quad \mathbf{x_2} \quad . . . \quad \mathbf{x_p})$$

Until one of the last sections in the notes, DeLury assumes $\mathbf{X}$ has full column rank. i.e that the columns of $\mathbf{X}$ are linearly independent.

The usual restricted regression question p2

A regression is a conditional mean $E(y|\mathbf{x}')$
A linear regression is $E(y|\mathbf{x}') = \mathbf{x}'\boldsymbol{\beta}$ where $\boldsymbol{\beta}' = (\beta_0 \quad \beta_1 \quad \beta_2 \quad . . . \quad \beta_p)$
The variance about regression is the conditional variance $Var(y|\mathbf{x}') = \sigma^2$

Assumptions about the sample p3

$E(y_\alpha|\mathbf{x_\alpha}') = \mathbf{x_\alpha}'\boldsymbol{\beta}$ or $y_\alpha = \mathbf{x_\alpha}'\boldsymbol{\beta} + \epsilon_\alpha$ with $E(\epsilon_\alpha) = 0$ and $E(\epsilon_\alpha \epsilon_\beta) = \delta_{\alpha\beta}\sigma^2$ so that $E(\boldsymbol{\epsilon}\boldsymbol{\epsilon}') = \sigma^2 \mathbf{I_{NxN}}$
where $\boldsymbol{\epsilon}' = (\epsilon_1 \quad \epsilon_2 \quad \epsilon_3 \quad . . . \epsilon_N)$ and $\mathbf{I_{NxN}}$ is an NxN identity matrix
$E(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta}$ or $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$

The fit is $Y = \mathbf{x}'\mathbf{b}$ and so $Y_\alpha = \mathbf{x_\alpha}'\mathbf{b}$ and $\mathbf{Y} = \mathbf{X}\mathbf{b}$ where $\mathbf{Y}' = (Y_1 \quad Y_2 \quad Y_3 \quad . . . Y_N)$

The principle of least squares p4

$E = S(y_\alpha - Y_\alpha)^2 = (\mathbf{y} - \mathbf{Y})'(\mathbf{y} - \mathbf{Y})$

DeLury proceeds using calculus

$$\frac{\partial \mathrm{E}}{\partial \mathrm{b_j}} = -2\mathrm{S}(\mathrm{y}_\alpha - \mathrm{Y}_\alpha)\frac{\partial \mathrm{Y}_\alpha}{\partial \mathrm{b_j}} = -2\mathrm{S}(\mathrm{y}_\alpha - \mathrm{Y}_\alpha)\mathrm{x}_{j\alpha} = -2(\mathbf{y}-\mathbf{Y})'\mathbf{x_j} \quad j=0,1,2...p$$

note: $\mathbf{x_j}$ is the jth column of $\mathbf{X}$

Collecting these p+1 equations as a row, we get: $(\mathbf{y}-\mathbf{Y})'\mathbf{X}=\mathbf{0}'$
Or writing as a column (or taking the transpose): $\mathbf{X}'(\mathbf{y}-\mathbf{Y})=\mathbf{0}$

and so $\mathbf{X}'\mathbf{Y}=\mathbf{X}'\mathbf{y}$ and since $\mathbf{Y}=\mathbf{X}\mathbf{b}$ $\mathbf{X}'\mathbf{X}\mathbf{b}=\mathbf{X}'\mathbf{y}$

and then $\mathbf{A}\mathbf{b}=\mathbf{g}$ where $\mathbf{A}=\mathbf{X}'\mathbf{X}$ and $\mathbf{g}=\mathbf{X}'\mathbf{y}$

so if $\mathbf{A}\mathbf{C}=\mathbf{C}\mathbf{A}=\mathbf{I}$, then: $\mathbf{b}=\mathbf{C}\mathbf{g}$

To avoid calculus, one can 'complete the square' [Fraser(1976) p378]:

$$\mathrm{E}=(\mathbf{y}-\mathbf{Y})'(\mathbf{y}-\mathbf{Y})=(\mathbf{y}-\mathbf{X}\mathbf{b})'(\mathbf{y}-\mathbf{X}\mathbf{b})$$

$$= (\mathbf{b}-(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y})'\mathbf{X}'\mathbf{X}(\mathbf{b}-(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y})+\mathbf{y}'\mathbf{y}-\mathbf{y}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

The positive definite quadratic form $\mathbf{z}'\mathbf{X}'\mathbf{X}\mathbf{z}$ is 0 only if $\mathbf{z}=0$.

<u>Best Linear Unbiased Estimates</u> p5
[Matrix notation enables a complete solution]

$\mathbf{b}=\mathbf{W}\mathbf{y}$ where $\mathbf{W}'=(\mathbf{w_0} \quad \mathbf{w_1} \quad \mathbf{w_2} \quad ...\mathbf{w_p})$ and $\mathrm{b_i}=\mathbf{w_i}'\mathbf{y}$

Using $\mathrm{E}\,\mathbf{b}=\boldsymbol{\beta}$ then $\mathrm{E}\,\mathbf{b}=\mathbf{W}\mathbf{X}\boldsymbol{\beta}=\boldsymbol{\beta}$ and so $\mathbf{W}\mathbf{X}=\mathbf{I}$
$\mathrm{Var}\,\mathbf{b}=\mathrm{Var}(\mathbf{W}\mathbf{y})=\sigma^2\mathbf{W}\mathbf{W}'$ [Fraser(1976) p200]

Using a 'complete the square' approach again [Fraser(1976) p386]:

$$\mathbf{W}\mathbf{W}'=(\mathbf{W}-(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')(\mathbf{W}-(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')'+(\mathbf{X}'\mathbf{X})^{-1}=\mathbf{C}\mathbf{C}'+(\mathbf{X}'\mathbf{X})^{-1}$$

$\mathbf{C}\mathbf{C}'$ is an inner product matrix [Fraser (1976) p520] and so is positive semi-definite.

[DeLury offers an approach via Lagrange Multipliers that establishes p+1 conditions that are consistent with the least squares solution]

<u>Variances and covariances of the b's</u> p6

$$\mathrm{Var}\,\mathbf{b}=(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\sigma^2\mathbf{I}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}=\sigma^2(\mathbf{X}'\mathbf{X})^{-1}$$

<u>Estimation of the error variance</u> p7

$$(\mathbf{y}-\mathbf{Y})'(\mathbf{y}-\mathbf{Y})=(\mathbf{y}-\mathbf{Y})'\mathbf{y}-(\mathbf{y}-\mathbf{Y})'\mathbf{Y}=(\mathbf{y}-\mathbf{Y})'\mathbf{y}-(\mathbf{y}-\mathbf{Y})'\mathbf{X}\mathbf{b}$$
$$=(\mathbf{y}-\mathbf{X}\mathbf{b})'\mathbf{y}-(\mathbf{y}-\mathbf{Y})'\mathbf{X}\mathbf{b}=\mathbf{y}'\mathbf{y}-\mathbf{b}'\mathbf{X}'\mathbf{y}-0=\mathbf{y}'\mathbf{y}-\mathbf{b}'\mathbf{g}$$

[the case $\boldsymbol{x_0}=\mathbf{1}$??]

Tests of significance p8

$\mathbf{b} \sim N(\boldsymbol{\beta}, \ \sigma^2(\mathbf{X}'\mathbf{X})^{-1})$ and so $\mathbf{b}-\boldsymbol{\beta} \sim N(\mathbf{0}, \ \sigma^2(\mathbf{X}'\mathbf{X})^{-1})$

accordingly $(\mathbf{b}-\boldsymbol{\beta})'(\mathbf{X}'\mathbf{X})(\mathbf{b}-\boldsymbol{\beta}) \sim \sigma^2\chi^2_{p+1}$ independent of $(\mathbf{y}-\mathbf{Y})'(\mathbf{y}-\mathbf{Y}) \sim \sigma^2\chi^2_{N-p-1}$

$$\mathbf{b}=\begin{pmatrix}\mathbf{b_1}\\ \mathbf{b_2}\end{pmatrix} \quad \mathbf{A}=\begin{pmatrix}\mathbf{A_{11}} & \mathbf{A_{12}}\\ \mathbf{A_{21}} & \mathbf{A_{22}}\end{pmatrix} \quad \mathbf{C}=\begin{pmatrix}\mathbf{C_{11}} & \mathbf{C_{12}}\\ \mathbf{C_{21}} & \mathbf{C_{22}}\end{pmatrix}$$

$\mathbf{b_2}-\boldsymbol{\beta_2} \sim N(\mathbf{0}, \ \sigma^2\mathbf{C_{22}})$ and $(\mathbf{b_2}-\boldsymbol{\beta_2})'(\mathbf{C_{22}^{-1}})(\mathbf{b_2}-\boldsymbol{\beta_2}) \sim \sigma^2\chi^2_{p-q}$

The "straight" regression line p9

$$\mathbf{X}=(\mathbf{1} \quad \mathbf{x}) \quad \mathbf{A}=\mathbf{X}'\mathbf{X}=\begin{pmatrix}\mathbf{1}'\\ \mathbf{x}'\end{pmatrix}(\mathbf{1} \quad \mathbf{x})=\begin{pmatrix}\mathbf{1}'\mathbf{1} & \mathbf{1}'\mathbf{x}\\ \mathbf{x}'\mathbf{1} & \mathbf{x}'\mathbf{x}\end{pmatrix} \text{ and } \mathbf{g}=\mathbf{X}'\mathbf{y}=\begin{pmatrix}\mathbf{1}'\mathbf{y}\\ \mathbf{x}'\mathbf{y}\end{pmatrix}$$

$$\mathbf{X}=(\mathbf{1} \quad \mathbf{x}-\bar{x}\mathbf{1}) \quad \mathbf{A}=\mathbf{X}'\mathbf{X}=\begin{pmatrix}\mathbf{1}'\mathbf{1} & \mathbf{0}\\ \mathbf{0} & (\mathbf{x}-\bar{x}\mathbf{1})'(\mathbf{x}-\bar{x}\mathbf{1})\end{pmatrix} \text{ and } \mathbf{g}=\mathbf{X}'\mathbf{y}=\begin{pmatrix}\mathbf{1}'\mathbf{y}\\ (\mathbf{x}-\bar{x}\mathbf{1})'\mathbf{y}\end{pmatrix} \quad \mathbf{b}=\begin{pmatrix}\bar{y}\\ b_1\end{pmatrix}$$

$$(\mathbf{y}-\mathbf{Y})'(\mathbf{y}-\mathbf{Y})=\mathbf{y}'\mathbf{y}-\mathbf{b}'\mathbf{g}=\mathbf{y}'\mathbf{y}-\bar{y}\mathbf{1}'\mathbf{y}-b_1(\mathbf{x}-\bar{x}\mathbf{1})'\mathbf{y}$$

$$\mathbf{z}=\mathbf{Q}'\mathbf{y} \quad \mathbf{Q}=\mathbf{D}^{-1/2}(\mathbf{1} \quad \mathbf{x}-\bar{x}\mathbf{1} \quad \mathbf{A}) \text{ where } \mathbf{A}_{\mathbf{Nx(N-2)}} \text{ and } \mathbf{D}=\operatorname{diag}\begin{pmatrix}\mathbf{1}'\mathbf{1}\\ (\mathbf{x}-\bar{x}\mathbf{1})'(\mathbf{x}-\bar{x}\mathbf{1})\\ \operatorname{diag}\mathbf{A}'\mathbf{A}\end{pmatrix}$$

$\mathbf{Q}'\mathbf{Q}=\mathbf{I}$ and so $\mathbf{A}'\mathbf{1}=\mathbf{0}_{\mathbf{(N-2)x1}}$ $\quad \mathbf{A}'(\mathbf{x}-\bar{x}\mathbf{1})=\mathbf{A}'\mathbf{x}=\mathbf{0}_{\mathbf{(N-2)x1}}$

$$\mathbf{z}=\begin{pmatrix}\mathbf{z_1}\\ \mathbf{z_2}\end{pmatrix} \text{ where } \mathbf{z_2}=\mathbf{A}'\mathbf{y} \text{ so } E(\mathbf{z_2})=\mathbf{A}'\mathbf{X}\boldsymbol{\beta}=\mathbf{A}'(\mathbf{1} \quad \mathbf{x})\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}=\mathbf{0}$$

$\operatorname{Var}\mathbf{z}=\mathbf{Q}\sigma^2\mathbf{I}\mathbf{Q}'=\sigma^2\mathbf{Q}\mathbf{Q}'=\sigma^2\mathbf{I}$ $\quad \mathbf{z} \sim N(E(\mathbf{z}),\sigma^2\mathbf{I})$ $\quad \mathbf{z_2}'\mathbf{z_2} \sim \sigma^2\chi^2_{N-2}$ $\quad \mathbf{z}'\mathbf{z}=\mathbf{y}\mathbf{Q}\mathbf{Q}'\mathbf{y}=\mathbf{y}'\mathbf{y}$

[add further separation of components of z]

Testing the assumption $Ey=\beta_0+\beta_1 x$ p11

$$\mathbf{y}=\begin{pmatrix}\mathbf{y_1}\\ \mathbf{y_2}\\ \cdot\\ \cdot\\ \cdot\\ \mathbf{y_k}\end{pmatrix} \quad \mathbf{X}=\begin{pmatrix}\mathbf{1_1} & x_1\mathbf{1_1}\\ \mathbf{1_2} & x_2\mathbf{1_2}\\ \cdot & \cdot\\ \cdot & \cdot\\ \cdot & \cdot\\ \mathbf{1_k} & x_k\mathbf{1_k}\end{pmatrix} \quad \mathbf{X}'\mathbf{y}=\begin{pmatrix}\sum_{i=1}^{k} T_i\\ \sum_{i=1}^{k} x_i T_i\end{pmatrix} \text{ where } T_i=\mathbf{1_i}'\mathbf{y_i}$$

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & x_k \end{pmatrix} = (\mathbf{1} \quad \mathbf{x}) \quad \text{and so } \mathbf{X}'\mathbf{y} = \begin{pmatrix} \mathbf{1}' \\ \mathbf{x}' \end{pmatrix} \mathbf{T} \quad \text{where } \mathbf{T} = \begin{pmatrix} T_1 \\ T_2 \\ \cdot \\ \cdot \\ \cdot \\ T_k \end{pmatrix}$$

$$\mathbf{E}'\mathbf{y} = \mathbf{T} \quad \text{and } \mathbf{E}(\mathbf{1} \quad \mathbf{x}) = \mathbf{X} \quad \text{where } \mathbf{E} = \begin{pmatrix} \mathbf{1_1} & \mathbf{0} & \mathbf{0} & . & . & . & \mathbf{0} \\ \mathbf{0} & \mathbf{1_2} & \mathbf{0} & . & . & . & \mathbf{0} \\ . & . & . & . & . & . & . \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & . & . & . & \mathbf{1_k} \end{pmatrix}_{Nxk}$$

$$\mathbf{X}'\mathbf{y} = \begin{pmatrix} \mathbf{1}' \\ \mathbf{x}' \end{pmatrix} \mathbf{E}'\mathbf{y} \quad \mathbf{X}'\mathbf{X} = \begin{pmatrix} \mathbf{1}' \\ \mathbf{x}' \end{pmatrix} \mathbf{E}'\mathbf{E}(\mathbf{1} \quad \mathbf{x}) = \begin{pmatrix} \mathbf{1}' \\ \mathbf{x}' \end{pmatrix} \text{diag}(\mathbf{n})(\mathbf{1} \quad \mathbf{x}) \quad \text{where } \mathbf{n}' = (n_1 \quad n_2 \quad . \quad . \quad . \quad n_k)$$

$$\mathbf{Q} = \mathbf{E}_{Nxk}(\mathbf{1} \quad \mathbf{x} - \bar{x}\,\mathbf{1} \quad \mathbf{A})_{kxk} \begin{pmatrix} \mathbf{Q_1} & \mathbf{0} & . & . & . & \mathbf{0} \\ \mathbf{0} & \mathbf{Q_2} & . & . & . & \mathbf{0} \\ . & . & . & . & . & . \\ \mathbf{0} & \mathbf{0} & . & . & . & \mathbf{Q_k} \end{pmatrix}_{Nx(N-k)} \qquad (\mathbf{Q_i})_{n_i x(n_i-1)} \text{ with } \mathbf{1_i}'\mathbf{Q_i} = \mathbf{0} \text{ and } \mathbf{Q_i}'\mathbf{Q_i} = \mathbf{I}$$

$\mathbf{z} = \mathbf{Q}'\mathbf{y} \quad \mathbf{z}' = (z_1 \quad z_2 \quad \mathbf{z_3}' \quad \mathbf{z_4}') \quad \mathbf{z_3} = \mathbf{A}'\mathbf{E}'\mathbf{y}$ and $\mathbf{z_4} = \text{diag}(\mathbf{Q_1}' \ldots \mathbf{Q_k}')\mathbf{y}$ [check this]

Orthogonal Functions p14

$Y_\alpha = \mathbf{x}'_\alpha \mathbf{b} = \mathbf{P}'_\alpha \mathbf{B}$ with $\mathbf{P}' = (\mathbf{P_1} \quad \mathbf{P_2} \quad . \quad . \quad . \quad \mathbf{P_N})$ so then $\mathbf{X} = \mathbf{P}'\mathbf{P}$ and $\mathbf{G} = \mathbf{P}'\mathbf{y}$

$\mathbf{P_\alpha} = \Lambda\,\mathbf{x_\alpha} \quad \mathbf{P} = \mathbf{X}\Lambda' \quad \mathbf{b} = \Lambda'\mathbf{B} \quad \mathbf{B} = (\Lambda')^{-1}\mathbf{b}$

choose $\mathbf{P}$ so that $\mathbf{P}'\mathbf{P} = \text{diag}(\mathbf{d}) = \mathbf{D}$ Λ may be lower triangular

$\mathbf{z} = \mathbf{Q}'\mathbf{y}$ where $\mathbf{Q} = \mathbf{D}^{-1/2}(\mathbf{P}, \mathbf{A})$ where $\mathbf{A}'\mathbf{P} = \mathbf{0}$

$\mathbf{z} = \begin{pmatrix} \mathbf{z_1} \\ \mathbf{z_2} \end{pmatrix}$ with $\mathbf{z_2} = \mathbf{A}'\mathbf{y}$ so that $E(\mathbf{z_2}) = \mathbf{A}'\mathbf{X}\,\boldsymbol{\beta} = \mathbf{A}'\mathbf{P}(\Lambda')^{-1}\boldsymbol{\beta} = \mathbf{0}$

$\mathbf{z_2}'\mathbf{z_2} \sim \sigma^2\chi^2_{N-p-1}$

Orthogonal Polynomials p18

$\mathbf{x}' = (1 \quad x \quad x^2 \quad x^3 \quad . \quad . \quad . \quad x^p)$ $\boldsymbol{\xi} = \Lambda\,\mathbf{x}$ when $\lambda_{ii} = 1$ $\boldsymbol{\xi}' = \boldsymbol{\xi}\,\text{diag}\,\boldsymbol{\lambda}(\mathbf{N})$

The Fundamental Distribution Theorem p21

$$\mathbf{x}=\begin{pmatrix}\mathbf{x}_1\\ \mathbf{x}_2\end{pmatrix} \quad \mathbf{Y}=\mathbf{x}'\mathbf{b} \quad \mathbf{Y}^1=\mathbf{x}_1'\mathbf{b}_1^1 \quad \mathbf{b}=\begin{pmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{pmatrix}$$

$$\begin{aligned} & (\mathbf{y}-\mathbf{Y}^1)'(\mathbf{y}-\mathbf{Y}^1)-(\mathbf{y}-\mathbf{Y})'(\mathbf{y}-\mathbf{Y}) \\ = \; & (\mathbf{Y}-\bar{y}\,\mathbf{1})'(\mathbf{Y}-\bar{y}\,\mathbf{1})-(\mathbf{Y}^1-\bar{y}\,\mathbf{1})'(\mathbf{Y}^1-\bar{y}\,\mathbf{1}) \\ = \; & (\mathbf{Y}-\mathbf{Y}^1)'(\mathbf{Y}-\mathbf{Y}^1) \end{aligned}$$

$$\begin{aligned} = \; & \mathbf{b}_2'\mathbf{C}_{22}^{-1}\mathbf{b}_2 \\ = \; & \mathbf{b}_2'\mathbf{g}_2^* \\ = \; & \mathbf{g}_2^{*\prime}\mathbf{C}_{22}\mathbf{g}_2^* \;\; \text{where}\, \mathbf{g}_2^*=\mathbf{C}_{22}^{-1}\mathbf{b}_2 \end{aligned}$$

$$\mathbf{P}=\begin{pmatrix}\mathbf{P}_1\\ \mathbf{P}_2\end{pmatrix} \quad \mathbf{Y}=\mathbf{P}'\mathbf{B} \quad \mathbf{Y}^1=\mathbf{P}_1'\mathbf{B}_1 \quad \mathbf{B}=\begin{pmatrix}\mathbf{B}_1\\ \mathbf{B}_1\end{pmatrix} \quad \mathbf{G}=\mathbf{P}'\mathbf{y}=\mathbf{D}\,\mathbf{B} \quad \mathbf{D}=\begin{pmatrix}\mathbf{D}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{D}_2\end{pmatrix}$$

$$(\mathbf{y}-\mathbf{Y})'(\mathbf{y}-\mathbf{Y})=\mathbf{y}'\mathbf{y}-\mathbf{B}'\mathbf{G}=\mathbf{y}'\mathbf{y}-\mathbf{B}_1'\mathbf{G}_1-\mathbf{B}_2'\mathbf{G}_2=\mathbf{y}'\mathbf{y}-\mathbf{B}'\mathbf{D}\,\mathbf{B}=\mathbf{y}'\mathbf{y}-\mathbf{B}_1'\mathbf{D}_1\mathbf{B}_1-\mathbf{B}_2'\mathbf{D}_2\mathbf{B}_2$$

$$(\mathbf{y}-\mathbf{Y}^1)'(\mathbf{y}-\mathbf{Y}^1)=\mathbf{y}'\mathbf{y}-\mathbf{B}_1'\mathbf{G}_1=\mathbf{y}'\mathbf{y}-\mathbf{B}_1'\mathbf{D}_1\mathbf{B}_1$$

$$(\mathbf{Y}-\mathbf{Y}^1)'(\mathbf{Y}-\mathbf{Y}^1)=(\mathbf{P}_2\mathbf{B}_2)'(\mathbf{P}_2\mathbf{B}_2)=\mathbf{B}_2'\mathbf{D}_2\mathbf{B}_2=\mathbf{B}_2'\mathbf{G}_2$$

$$\mathbf{A_p}\mathbf{B_p}=\mathbf{G_p} \quad \mathbf{A_p}=\mathbf{P}'\mathbf{P}=\Lambda\,\mathbf{X}'\mathbf{X}\,\Lambda \quad \mathbf{G_p}=\mathbf{P}'\mathbf{y}=\Lambda\,\mathbf{X}'\mathbf{y}=\Lambda\,\mathbf{g_p} \quad \text{[more steps needed?]}$$

<u>The solution of normal equations</u> p25

$$\left(\begin{pmatrix}\mathbf{x_0}'\\ \mathbf{x_1}'\\ \mathbf{x_2}'\\ \mathbf{x_3}'\\ \mathbf{y}'\end{pmatrix}\begin{pmatrix}\mathbf{x_0} & \mathbf{x_1} & \mathbf{x_2} & \mathbf{x_3} & \mathbf{y}\end{pmatrix}\middle|\begin{matrix}\mathbf{I_{4x4}}\\ \mathbf{0}'\end{matrix}\right)=\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} & g_0 & 1 & 0 & 0 & 0\\ a_{10} & a_{11} & a_{12} & a_{13} & g_1 & 0 & 1 & 0 & 0\\ a_{20} & a_{21} & a_{22} & a_{23} & g_2 & 0 & 0 & 1 & 0\\ a_{30} & a_{31} & a_{32} & a_{33} & g_3 & 0 & 0 & 0 & 1\\ g_0 & g_1 & g_2 & g_3 & \mathbf{y}'\mathbf{y} & 0 & 0 & 0 & 0\end{pmatrix}$$

$$\begin{pmatrix}1/N & \mathbf{0}' & 0\\ -\bar{\mathbf{x}} & \mathbf{I} & \mathbf{0}\\ -\bar{y} & \mathbf{0}' & 1\end{pmatrix}\begin{pmatrix}N & N\bar{\mathbf{x}} & \mathbf{1}'\mathbf{y}\\ N\bar{\mathbf{x}} & \mathbf{X}'\mathbf{X} & \mathbf{X}'\mathbf{y}\\ \mathbf{y}'\mathbf{1} & \mathbf{y}'\mathbf{X} & \mathbf{y}'\mathbf{y}\end{pmatrix}$$

$$=\begin{pmatrix}1 & \bar{\mathbf{x}}' & \bar{y}\\ \mathbf{0} & \mathbf{X}'\mathbf{X}-N\,\bar{\mathbf{x}}\bar{\mathbf{x}}' & \mathbf{X}'\mathbf{y}-N\,\bar{y}\,\bar{\mathbf{x}}\\ 0 & \mathbf{y}'\mathbf{X}-\mathbf{N}\,\bar{\mathbf{x}}'\bar{y} & \mathbf{y}'\mathbf{y}-N\,\bar{y}^2\end{pmatrix}=\begin{pmatrix}1 & \bar{\mathbf{x}}' & \bar{y}\\ \mathbf{0} & (\mathbf{C_N}\mathbf{X})'(\mathbf{C_N}\mathbf{X}) & (\mathbf{C_N}\mathbf{X})'(\mathbf{C_N}\mathbf{y})\\ 0 & (\mathbf{C_N}\mathbf{y})'(\mathbf{C_N}\mathbf{X}) & (\mathbf{C_N}\mathbf{y})'(\mathbf{C_N}\mathbf{y})\end{pmatrix}$$

$$\mathbf{C_N}=\mathbf{I}-\bar{\mathbf{J}}_\mathbf{N} \quad \bar{\mathbf{J}}_\mathbf{N}=\mathbf{1}\mathbf{1}'/N$$

$$\begin{pmatrix} (\mathbf{X}_1'\mathbf{X}_1)^{-1} & \mathbf{0} & \mathbf{0} \\ -\mathbf{X}_2'\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1} & \mathbf{I} & \mathbf{0} \\ -\mathbf{y}'\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1} & \mathbf{0}' & 1 \end{pmatrix} \begin{pmatrix} \mathbf{X}_1'\mathbf{X}_1 & \mathbf{X}_1'\mathbf{X}_2 & \mathbf{X}_1'\mathbf{y} \\ \mathbf{X}_2'\mathbf{X}_1 & \mathbf{X}_2'\mathbf{X}_2 & \mathbf{X}_2'\mathbf{y} \\ \mathbf{y}'\mathbf{X}_1 & \mathbf{y}'\mathbf{X}_2 & \mathbf{y}'\mathbf{y} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{I} & \mathbf{A}_{12.1} & \mathbf{b}_1^1 \\ \mathbf{0} & \mathbf{X}_{2.1}'\mathbf{X}_{2.1} & \mathbf{X}_{2.1}'\mathbf{y} \\ 0 & \mathbf{y}'\mathbf{X}_{2.1} & \mathbf{y}'\mathbf{P}_1\mathbf{y} \end{pmatrix} \quad \text{where} \quad \mathbf{A}_{12.1} = (\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2 \ \text{ and } \ \mathbf{X}_{2.1} = \mathbf{X}_2 - \mathbf{X}_1\mathbf{A}_{12.1}$$

$$= \begin{pmatrix} \mathbf{I} & -\mathbf{A}_{12.1}(\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{y}'\mathbf{X}_{2.1}(\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{A}_{12.1} & \mathbf{b}_1^1 \\ \mathbf{0} & \mathbf{X}_{2.1}'\mathbf{X}_{2.1} & \mathbf{X}_{2.1}'\mathbf{y} \\ \mathbf{0} & \mathbf{y}'\mathbf{X}_{2.1} & \mathbf{y}'\mathbf{P}_1\mathbf{y} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{b}_1 = \mathbf{b}_1^1 - \mathbf{A}_{12.1}\mathbf{b}_2 \\ \mathbf{0} & \mathbf{I} & \mathbf{b}_2 = (\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1}\mathbf{X}_{2.1}'\mathbf{y} \\ \mathbf{0} & \mathbf{0} & \mathbf{y}'\mathbf{P}_2\mathbf{y} \end{pmatrix}$$

$$\begin{pmatrix} (\mathbf{X}_1'\mathbf{X}_1)^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{X}_2'\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ -\mathbf{X}_3'\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ -\mathbf{y}'\mathbf{X}_1(\mathbf{X}_1'\mathbf{X}_1)^{-1} & \mathbf{0}' & \mathbf{0}' & 1 \end{pmatrix} \begin{pmatrix} \mathbf{X}_1'\mathbf{X}_1 & \mathbf{X}_1'\mathbf{X}_2 & \mathbf{X}_1'\mathbf{X}_3 & \mathbf{X}_1'\mathbf{y} \\ \mathbf{X}_2'\mathbf{X}_1 & \mathbf{X}_2'\mathbf{X}_2 & \mathbf{X}_2'\mathbf{X}_3 & \mathbf{X}_2'\mathbf{y} \\ \mathbf{X}_3'\mathbf{X}_1 & \mathbf{X}_3'\mathbf{X}_2 & \mathbf{X}_3'\mathbf{X}_3 & \mathbf{X}_3'\mathbf{y} \\ \mathbf{y}'\mathbf{X}_1 & \mathbf{y}'\mathbf{X}_2 & \mathbf{y}'\mathbf{X}_3 & \mathbf{y}'\mathbf{y} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{I} & \mathbf{A}_{12.1} & \mathbf{A}_{13.1} & \mathbf{b}_1^{(1)} \\ \mathbf{0} & \mathbf{X}_{2.1}'\mathbf{X}_{2.1} & \mathbf{X}_{2.1}'\mathbf{X}_{3.1} & \mathbf{X}_{2.1}'\mathbf{y} \\ \mathbf{0} & \mathbf{X}_{3.1}'\mathbf{X}_{2.1} & \mathbf{X}_{3.1}'\mathbf{X}_{3.1} & \mathbf{X}_{3.1}'\mathbf{y} \\ \mathbf{0} & \mathbf{y}'\mathbf{X}_{2.1} & \mathbf{y}'\mathbf{X}_{3.1} & \mathbf{y}'\mathbf{P}_1\mathbf{y} \end{pmatrix} \qquad \begin{matrix} \mathbf{A}_{12.1}=(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_2 \text{ and } \mathbf{X}_{2.1}=\mathbf{X}_2-\mathbf{X}_1\mathbf{A}_{12.1} \\ \mathbf{A}_{13.1}=(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{X}_1'\mathbf{X}_3 \text{ and } \mathbf{X}_{3.1}=\mathbf{X}_3-\mathbf{X}_1\mathbf{A}_{13.1} \end{matrix}$$

$$\begin{pmatrix} \mathbf{I} & -\mathbf{A}_{12.1}(\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{X}_{3.1}'\mathbf{X}_{2.1}(\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1} & \mathbf{I} & \mathbf{0} \\ \mathbf{0}' & -\mathbf{y}'\mathbf{X}_{2.1}(\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1} & \mathbf{0}' & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{A}_{12.1} & \mathbf{A}_{13.1} & \mathbf{b}_1^{(1)} \\ \mathbf{0} & \mathbf{X}_{2.1}'\mathbf{X}_{2.1} & \mathbf{X}_{2.1}'\mathbf{X}_{3.1} & \mathbf{X}_{2.1}'\mathbf{y} \\ \mathbf{0} & \mathbf{X}_{3.1}'\mathbf{X}_{2.1} & \mathbf{X}_{3.1}'\mathbf{X}_{3.1} & \mathbf{X}_{3.1}'\mathbf{y} \\ \mathbf{0} & \mathbf{y}'\mathbf{X}_{2.1} & \mathbf{y}'\mathbf{X}_{3.1} & \mathbf{y}'\mathbf{P}_1\mathbf{y} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{A}_{13.2} & \mathbf{b}_1^{(2)} \\ \mathbf{0} & \mathbf{I} & \mathbf{A}_{23.2} & \mathbf{b}_2^{(2)} \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_{3.2}'\mathbf{X}_{3.2} & \mathbf{X}_{3.2}'\mathbf{y} \\ \mathbf{0} & \mathbf{0} & \mathbf{y}'\mathbf{X}_{3.2} & \mathbf{y}'\mathbf{P}_2\mathbf{y} \end{pmatrix} \qquad \begin{matrix} \mathbf{A}_{23.2}=(\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1}\mathbf{X}_{2.1}'\mathbf{X}_{3.1} \\ \mathbf{A}_{13.2}=\mathbf{A}_{13.1}-\mathbf{A}_{12.1}\mathbf{A}_{23.2} \\ \mathbf{X}_{3.2}=\mathbf{X}_{3.1}-\mathbf{X}_{2.1}\mathbf{A}_{23.2} \end{matrix} \qquad \begin{matrix} \mathbf{b}_2^{(2)}=(\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1}\mathbf{X}_{2.1}'\mathbf{y} \\ \mathbf{b}_1^{(2)}=\mathbf{b}_1^{(1)}-\mathbf{A}_{12.1}\mathbf{b}_2^{(2)} \end{matrix}$$

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} & -\mathbf{A}_{13.2}(\mathbf{X}_{3.2}'\mathbf{X}_{3.2})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & -\mathbf{A}_{23.2}(\mathbf{X}_{3.2}'\mathbf{X}_{3.2})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\mathbf{X}_{3.2}'\mathbf{X}_{3.2})^{-1} & \mathbf{0} \\ \mathbf{0}' & \mathbf{0}' & -\mathbf{y}'\mathbf{X}_{3.2}(\mathbf{X}_{3.2}'\mathbf{X}_{3.2})^{-1} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{A}_{13.2} & \mathbf{b}_1^{(2)} \\ \mathbf{0} & \mathbf{I} & \mathbf{A}_{23.2} & \mathbf{b}_2^{(2)} \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_{3.2}'\mathbf{X}_{3.2} & \mathbf{X}_{3.2}'\mathbf{y} \\ \mathbf{0} & \mathbf{0} & \mathbf{y}'\mathbf{X}_{3.2} & \mathbf{y}'\mathbf{P}_2\mathbf{y} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{b}_1 \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{b}_2 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{b}_3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{y}'\mathbf{P}_3\mathbf{y} \end{pmatrix} \qquad \begin{matrix} \mathbf{b}_1=\mathbf{b}_1^{(2)}-\mathbf{A}_{13.2}\mathbf{b}_3 \\ \mathbf{b}_2=\mathbf{b}_2^{(2)}-\mathbf{A}_{23.2}\mathbf{b}_3 \\ \mathbf{b}_3=(\mathbf{X}_{3.2}'\mathbf{X}_{3.2})^{-1}\mathbf{X}_{3.2}'\mathbf{y} \end{matrix}$$

$$\begin{pmatrix} \mathbf{I} & -\mathbf{A}_{12.1}(\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{A}_{23.2}' & \mathbf{I} \end{pmatrix} \begin{pmatrix} (\mathbf{X}_1'\mathbf{X}_1)^{-1}=\mathbf{C}_{11}^{(1)} & \mathbf{0} & \mathbf{0} \\ -\mathbf{A}_{12.1}' & \mathbf{I} & \mathbf{0} \\ -\mathbf{A}_{13.1}' & \mathbf{0} & \mathbf{I} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{C}_{11}^{(2)}=\mathbf{C}_{11}^{(1)}+\mathbf{A}_{12.1}(\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1}\mathbf{A}_{12.1}' & -\mathbf{A}_{12.1}(\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1} & \mathbf{0} \\ -(\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1}\mathbf{A}_{12.1}' & \mathbf{C}_{22}^{(2)}=(\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1} & \mathbf{0} \\ -\mathbf{A}_{13.2}' & -\mathbf{A}_{23.2}' & \mathbf{I} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} & -\mathbf{A}_{13.2}(\mathbf{X}_{3.2}'\mathbf{X}_{3.2})^{-1} \\ \mathbf{0} & \mathbf{I} & -\mathbf{A}_{23.2}(\mathbf{X}_{3.2}'\mathbf{X}_{3.2})^{-1} \\ \mathbf{0} & \mathbf{0} & (\mathbf{X}_{3.2}'\mathbf{X}_{3.2})^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{C}_{11}^{(2)} & -\mathbf{A}_{12.1}(\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1} & \mathbf{0} \\ -(\mathbf{X}_{2.1}'\mathbf{X}_{2.1})^{-1}\mathbf{A}_{12.1}' & \mathbf{C}_{22}^{(2)} & \mathbf{0} \\ -\mathbf{A}_{13.2}' & -\mathbf{A}_{23.2}' & \mathbf{I} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{C}_{11}=\mathbf{C}_{11}^{(2)}+\mathbf{A}_{13.2}(\mathbf{X}_{3.2}'\mathbf{X}_{3.2})^{-1}\mathbf{A}_{13.2}' & \mathbf{C}_{12}^{(2)}+\mathbf{A}_{13.2}(\mathbf{X}_{3.2}'\mathbf{X}_{3.2})^{-1}\mathbf{A}_{23.2}' & -\mathbf{A}_{13.2}(\mathbf{X}_{3.2}'\mathbf{X}_{3.2})^{-1} \\ \mathbf{C}_{12}^{(2)}{}'+\mathbf{A}_{23.2}(\mathbf{X}_{3.2}'\mathbf{X}_{3.2})^{-1}\mathbf{A}_{13.2}' & \mathbf{C}_{22}=\mathbf{C}_{22}^{(2)}+\mathbf{A}_{23.2}(\mathbf{X}_{3.2}'\mathbf{X}_{3.2})^{-1}\mathbf{A}_{23.2}' & -\mathbf{A}_{23.2}(\mathbf{X}_{3.2}'\mathbf{X}_{3.2})^{-1} \\ -(\mathbf{X}_{3.2}'\mathbf{X}_{3.2})^{-1}\mathbf{A}_{13.2}' & -(\mathbf{X}_{3.2}'\mathbf{X}_{3.2})^{-1}\mathbf{A}_{23.2}' & \mathbf{C}_{33}=(\mathbf{X}_{3.2}'\mathbf{X}_{3.2})^{-1} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{I} & \mathbf{0} & -\mathbf{A}_{13.2}\mathbf{C}_{33} \\ \mathbf{0} & \mathbf{I} & -\mathbf{A}_{23.2}\mathbf{C}_{33} \\ \mathbf{0} & \mathbf{0} & \mathbf{C}_{33} \end{pmatrix} \begin{pmatrix} \mathbf{C}_{11}^{(2)} & -\mathbf{A}_{12.1}\mathbf{C}_{22}^{(2)} & \mathbf{0} \\ -\mathbf{C}_{22}^{(2)}\mathbf{A}_{12.1}' & \mathbf{C}_{22}^{(2)} & \mathbf{0} \\ -\mathbf{A}_{13.2}' & -\mathbf{A}_{23.2}' & \mathbf{I} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{C}_{11}=\mathbf{C}_{11}^{(2)}+\mathbf{A}_{13.2}\mathbf{C}_{33}\mathbf{A}_{13.2}' & \mathbf{C}_{12}^{(2)}+\mathbf{A}_{13.2}\mathbf{C}_{33}\mathbf{A}_{23.2}' & -\mathbf{A}_{13.2}\mathbf{C}_{33} \\ \mathbf{C}_{12}^{(2)}{}'+\mathbf{A}_{23.2}\mathbf{C}_{33}\mathbf{A}_{13.2}' & \mathbf{C}_{22}=\mathbf{C}_{22}^{(2)}+\mathbf{A}_{23.2}\mathbf{C}_{33}\mathbf{A}_{23.2}' & -\mathbf{A}_{23.2}\mathbf{C}_{33} \\ -\mathbf{C}_{33}\mathbf{A}_{13.2}' & -\mathbf{C}_{33}\mathbf{A}_{23.2}' & \mathbf{C}_{33} \end{pmatrix}$$

Extension of the use of indicator variables

$$\begin{pmatrix} N & n_1 & n_2 & n_3 & . & . & . & n_k & 0 & g_0 \\ n_1 & n_1 & 0 & 0 & . & . & . & 0 & 1 & g_1 \\ n_2 & 0 & n_2 & 0 & . & . & . & 0 & 1 & g_2 \\ n_3 & 0 & 0 & n_3 & . & . & . & 0 & 1 & g_3 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ n_k & 0 & 0 & 0 & . & . & . & n_k & 1 & g_k \\ 0 & 1 & 1 & 1 & . & . & . & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{1}{kn_1} & \frac{1}{kn_2} & \frac{1}{kn_3} & . & . & . & \frac{1}{kn_k} & -\frac{1}{k} \\ 0 & \frac{1}{n_1} & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & \frac{1}{n_2} & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{n_3} & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & \frac{1}{n_k} & 0 \\ -\frac{1}{k} & \frac{1}{k} & \frac{1}{k} & \frac{1}{k} & . & . & . & \frac{1}{k} & 0 \end{pmatrix} \begin{pmatrix} N & n_1 & n_2 & n_3 & . & . & . & n_k & 0 & g_0 \\ n_1 & n_1 & 0 & 0 & . & . & . & 0 & 1 & g_1 \\ n_2 & 0 & n_2 & 0 & . & . & . & 0 & 1 & g_2 \\ n_3 & 0 & 0 & n_3 & . & . & . & 0 & 1 & g_3 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ n_k & 0 & 0 & 0 & . & . & . & n_k & 1 & g_k \\ 0 & 1 & 1 & 1 & . & . & . & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & . & . & . & 0 & \frac{1}{k}\sum\frac{1}{n_j} & \frac{1}{k}\sum\frac{g_j}{n_j} \\
1 & 1 & 0 & 0 & . & . & . & 0 & \frac{1}{n_1} & \frac{g_1}{n_1} \\
1 & 0 & 1 & 0 & . & . & . & 0 & \frac{1}{n_2} & \frac{g_2}{n_2} \\
1 & 0 & 0 & 1 & . & . & . & 0 & \frac{1}{n_3} & \frac{g_3}{n_3} \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
1 & 0 & 0 & 0 & . & . & . & 1 & \frac{1}{n_k} & \frac{g_k}{n_k} \\
0 & 0 & 0 & 0 & . & . & . & 0 & 1 & \frac{1}{k}(\sum g_j - g_0)
\end{pmatrix}$$

$$\begin{pmatrix}
1 & 0 & 0 & 0 & . & . & . & 0 & -\frac{1}{k}\sum\frac{1}{n_j} \\
-1 & 1 & 0 & 0 & . & . & . & 0 & -(\frac{1}{n_1}-\frac{1}{k}\sum\frac{1}{n_j}) \\
-1 & 0 & 1 & 0 & . & . & . & 0 & -(\frac{1}{n_2}-\frac{1}{k}\sum\frac{1}{n_j}) \\
-1 & 0 & 0 & 1 & . & . & . & 0 & -(\frac{1}{n_3}-\frac{1}{k}\sum\frac{1}{n_j}) \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
-1 & 0 & 0 & 0 & . & . & . & 1 & -(\frac{1}{n_k}-\frac{1}{k}\sum\frac{1}{n_j}) \\
0 & 0 & 0 & 0 & . & . & . & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 & . & . & . & 0 & \frac{1}{k}\sum\frac{1}{n_j} & \frac{1}{k}\sum\frac{g_j}{n_j} \\
1 & 1 & 0 & 0 & . & . & . & 0 & \frac{1}{n_1} & \frac{g_1}{n_1} \\
1 & 0 & 1 & 0 & . & . & . & 0 & \frac{1}{n_2} & \frac{g_2}{n_2} \\
1 & 0 & 0 & 1 & . & . & . & 0 & \frac{1}{n_3} & \frac{g_3}{n_3} \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
1 & 0 & 0 & 0 & . & . & . & 1 & \frac{1}{n_k} & \frac{g_k}{n_k} \\
0 & 0 & 0 & 0 & . & . & . & 0 & 1 & \frac{1}{k}(\sum g_j - g_0)
\end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & . & . & . & 0 & 0 & \frac{1}{k}\sum\frac{g_j}{n_j}-\frac{\lambda}{k}\sum\frac{1}{n_j} \\ 1 & 1 & 0 & 0 & . & . & . & 0 & 0 & \frac{g_1}{n_1}-\frac{1}{k}\sum\frac{g_j}{n_j}-\lambda(\frac{1}{n_1}-\frac{1}{k}\sum\frac{1}{n_j}) \\ 1 & 0 & 1 & 0 & . & . & . & 0 & 0 & \frac{g_2}{n_2}-\frac{1}{k}\sum\frac{g_j}{n_j}-\lambda(\frac{1}{n_2}-\frac{1}{k}\sum\frac{1}{n_j}) \\ 1 & 0 & 0 & 1 & . & . & . & 0 & 0 & \frac{g_3}{n_3}-\frac{1}{k}\sum\frac{g_j}{n_j}-\lambda(\frac{1}{n_3}-\frac{1}{k}\sum\frac{1}{n_j}) \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ 1 & 0 & 0 & 0 & . & . & . & 1 & 0 & \frac{g_k}{n_k}-\frac{1}{k}\sum\frac{g_j}{n_j}-\lambda(\frac{1}{n_k}-\frac{1}{k}\sum\frac{1}{n_j}) \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 1 & \frac{1}{k}(\sum g_j-g_0) \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & . & . & . & 0 & -\frac{1}{k}\sum\frac{1}{n_j} \\ -1 & 1 & 0 & 0 & . & . & . & 0 & -(\frac{1}{n_1}-\frac{1}{k}\sum\frac{1}{n_j}) \\ -1 & 0 & 1 & 0 & . & . & . & 0 & -(\frac{1}{n_2}-\frac{1}{k}\sum\frac{1}{n_j}) \\ -1 & 0 & 0 & 1 & . & . & . & 0 & -(\frac{1}{n_3}-\frac{1}{k}\sum\frac{1}{n_j}) \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ -1 & 0 & 0 & 0 & . & . & . & 1 & -(\frac{1}{n_k}-\frac{1}{k}\sum\frac{1}{n_j}) \\ 0 & 0 & 0 & 0 & . & . & . & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{kn_1} & \frac{1}{kn_2} & \frac{1}{kn_3} & . & . & . & \frac{1}{kn_k} & -\frac{1}{k} \\ 0 & \frac{1}{n_1} & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & \frac{1}{n_2} & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{n_3} & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & \frac{1}{n_k} & 0 \\ -\frac{1}{k} & \frac{1}{k} & \frac{1}{k} & \frac{1}{k} & . & . & . & \frac{1}{k} & 0 \end{pmatrix}$$

$$
\begin{pmatrix}
\frac{1}{k^2}\sum\frac{1}{n_j} & \frac{1}{kn_1}-\frac{1}{k^2}\sum\frac{1}{n_j} & \frac{1}{kn_2}-\frac{1}{k^2}\sum\frac{1}{n_j} & \cdots & \frac{1}{kn_k}-\frac{1}{k^2}\sum\frac{1}{n_j} & -\frac{1}{k} \\
\frac{1}{kn_1}-\frac{1}{k^2}\sum\frac{1}{n_j} & \frac{1}{n_1}(1-\frac{2}{k})+\frac{1}{k^2}\sum\frac{1}{n_j} & . & \cdots & . & \frac{1}{k} \\
\frac{1}{kn_2}-\frac{1}{k^2}\sum\frac{1}{n_j} & -\frac{1}{k}(\frac{1}{n_1}+\frac{1}{n_2})+\frac{1}{k^2}\sum\frac{1}{n_j} & \frac{1}{n_2}(1-\frac{2}{k})+\frac{1}{k^2}\sum\frac{1}{n_j} & \cdots & . & \frac{1}{k} \\
. & . & . & \cdots & . & . \\
. & . & . & \cdots & . & . \\
\frac{1}{kn_k}-\frac{1}{k^2}\sum\frac{1}{n_j} & -\frac{1}{k}(\frac{1}{n_1}+\frac{1}{n_k})+\frac{1}{k^2}\sum\frac{1}{n_j} & . & \cdots & \frac{1}{n_k}(1-\frac{2}{k})+\frac{1}{k^2}\sum\frac{1}{n_j} & \frac{1}{k} \\
-\frac{1}{k} & \frac{1}{k} & \frac{1}{k} & \cdots & \frac{1}{k} & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
N & n_{1.} & n_{2.} & n_{r.} & n_{.1} & n_{.2} & n_{.c} & n_{11} & n_{12} & n_{1c} & n_{21} & n_{22} & n_{2c} & n_{r1} & n_{r2} & n_{rc} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
n_{1.} & n_{1.} & 0 & 0 & n_{11} & n_{12} & n_{1c} & n_{11} & n_{12} & n_{1c} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
n_{2.} & 0 & n_{2.} & 0 & n_{21} & n_{22} & n_{2c} & 0 & 0 & 0 & n_{21} & n_{22} & n_{2c} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
n_{r.} & 0 & 0 & n_{r.} & n_{r1} & n_{r2} & n_{rc} & 0 & 0 & 0 & 0 & 0 & 0 & n_{r1} & n_{r2} & n_{rc} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
n_{.1} & n_{11} & n_{21} & n_{r1} & n_{.1} & 0 & 0 & n_{11} & 0 & 0 & n_{21} & 0 & 0 & n_{r1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
n_{.2} & n_{12} & n_{12} & n_{r2} & 0 & n_{.2} & 0 & 0 & n_{12} & 0 & 0 & n_{22} & 0 & 0 & n_{r2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
n_{.c} & n_{1c} & n_{2c} & n_{rc} & 0 & 0 & n_{.c} & 0 & 0 & n_{1c} & 0 & 0 & n_{2c} & 0 & 0 & n_{rc} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
n_{11} & n_{11} & 0 & 0 & n_{11} & 0 & 0 & n_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
n_{12} & n_{12} & 0 & 0 & 0 & n_{12} & 0 & 0 & n_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
n_{1c} & n_{1c} & 0 & 0 & 0 & 0 & n_{1c} & 0 & 0 & n_{1c} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
n_{21} & 0 & n_{21} & 0 & n_{21} & 0 & 0 & 0 & 0 & 0 & n_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
n_{22} & 0 & n_{22} & 0 & 0 & n_{22} & 0 & 0 & 0 & 0 & 0 & n_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
n_{2c} & 0 & n_{2c} & 0 & 0 & 0 & n_{2c} & 0 & 0 & 0 & 0 & 0 & n_{2c} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
n_{r1} & 0 & 0 & n_{r1} & n_{r1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_{r1} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
n_{r2} & 0 & 0 & n_{r2} & 0 & n_{r2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_{r2} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
n_{rc} & 0 & 0 & n_{rc} & 0 & 0 & n_{rc} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n_{rc} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

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\begin{pmatrix}
0&0&0&0&0&0&0&\frac{1}{rcn_{11}}&\frac{1}{rcn_{12}}&\frac{1}{rcn_{1c}}&\frac{1}{rcn_{21}}&\frac{1}{rcn_{22}}&\frac{1}{rcn_{2c}}&\frac{1}{rcn_{r1}}&\frac{1}{rcn_{r2}}&\frac{1}{rcn_{rc}}&-\frac{1}{r}&-\frac{1}{c}&-\frac{1}{rc}&-\frac{1}{rc}&-\frac{1}{rc}&0&0&0\\
0&0&0&0&0&0&0&\frac{1}{cn_{11}}&\frac{1}{cn_{12}}&\frac{1}{cn_{1c}}&0&0&0&0&0&0&0&-\frac{1}{c}&-\frac{1}{c}&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&\frac{1}{cn_{21}}&\frac{1}{cn_{22}}&\frac{1}{cn_{2c}}&0&0&0&0&-\frac{1}{c}&0&-\frac{1}{c}&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&\frac{1}{cn_{r1}}&\frac{1}{cn_{r2}}&\frac{1}{cn_{rc}}&0&-\frac{1}{c}&0&0&-\frac{1}{c}&0&0&0\\
0&0&0&0&0&0&0&\frac{1}{rn_{11}}&0&0&\frac{1}{rn_{21}}&0&0&\frac{1}{rn_{r1}}&0&0&-\frac{1}{r}&0&0&0&0&-\frac{1}{r}&0&0\\
0&0&0&0&0&0&0&0&\frac{1}{rn_{12}}&0&0&\frac{1}{rn_{22}}&0&0&\frac{1}{rn_{r2}}&0&-\frac{1}{r}&0&0&0&0&0&-\frac{1}{r}&0\\
0&0&0&0&0&0&0&0&0&\frac{1}{rn_{1c}}&0&0&\frac{1}{rn_{2c}}&0&0&\frac{1}{rn_{rc}}&-\frac{1}{r}&0&0&0&0&0&0&-\frac{1}{r}\\
0&0&0&0&0&0&0&1/n_{11}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1/n_{12}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1/n_{1c}&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1/n_{21}&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1/n_{22}&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1/n_{2c}&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1/n_{r1}&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1/n_{r2}&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1/n_{rc}&0&0&0&0&0&0&0&0\\
-1/r&1/r&1/r&1/r&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
-1/c&0&0&0&1/c&1/c&1/c&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-1/c&0&0&0&0&0&1/c&1/c&1/c&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1/c&0&0&0&0&0&0&0&1/c&1/c&1/c&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1/c&0&0&0&0&0&0&0&0&0&1/c&1/c&1/c&0&0&0&0&0&0&0&0\\
0&0&0&0&-1/r&0&0&1/r&0&0&1/r&0&0&1/r&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1/r&0&0&1/r&0&0&1/r&0&0&1/r&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1/r&0&0&1/r&0&0&1/r&0&0&0&1/r&0&0&0&0&0&0&1
\end{pmatrix}
$$

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\left(\begin{array}{cccccccccccccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{rc}\sum\frac{1}{n_{1j}} & \frac{1}{rc}\sum\frac{1}{n_{2j}} & \frac{1}{rc}\sum\frac{1}{n_{rj}} & \frac{1}{rc}\sum\frac{1}{n_{i1}} & \frac{1}{rc}\sum\frac{1}{n_{i2}} & \frac{1}{rc}\sum\frac{1}{n_{ic}} \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{c}\sum\frac{1}{n_{1j}} & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{c}\sum\frac{1}{n_{2j}} & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{c}\sum\frac{1}{n_{rj}} & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{r}\sum\frac{1}{n_{i1}} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{r}\sum\frac{1}{n_{i2}} & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{r}\sum\frac{1}{n_{ic}} \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/n_{11} & 0 & 0 & 1/n_{11} & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/n_{12} & 0 & 0 & 0 & 1/n_{12} & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/n_{1c} & 0 & 0 & 0 & 0 & 1/n_{1c} \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/n_{21} & 0 & 1/n_{21} & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/n_{22} & 0 & 0 & 1/n_{22} & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/n_{2c} & 0 & 0 & 0 & 1/n_{2c} \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/n_{r1} & 1/n_{r1} & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1/n_{r2} & 0 & 1/n_{r2} & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1/n_{rc} & 0 & 0 & 1/n_{rc} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/c & 0 & 1 & 0 & 0 & 1/c & 1/c & 1/c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/c & 0 & 0 & 1 & 0 & 1/c & 1/c & 1/c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/c & 0 & 0 & 0 & 1 & 1/c & 1/c & 1/c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/r & 1/r & 1/r & 1/r & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/r & 1/r & 1/r & 1/r & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/r & 1/r & 1/r & 1/r & 0 & 0 & 1
\end{array}\right)
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\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/c & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/c & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/c & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/r & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/r & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/r & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
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\left(\begin{array}{cccccccccccccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{rc}\sum\frac{1}{n_{1j}} & \frac{1}{rc}\sum\frac{1}{n_{2j}} & \frac{1}{rc}\sum\frac{1}{n_{rj}} & \frac{1}{rc}\sum\frac{1}{n_{i1}} & \frac{1}{rc}\sum\frac{1}{n_{i2}} & \frac{1}{rc}\sum\frac{1}{n_{ic}} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{c}(1-\frac{1}{r})\sum\frac{1}{n_{1j}} & -\frac{1}{rc}\sum\frac{1}{n_{2j}} & -\frac{1}{rc}\sum\frac{1}{n_{rj}} & -\frac{1}{rc}\sum\frac{1}{n_{i1}} & -\frac{1}{rc}\sum\frac{1}{n_{i2}} & -\frac{1}{rc}\sum\frac{1}{n_{ic}} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{rc}\sum\frac{1}{n_{1j}} & \frac{1}{c}(1-\frac{1}{r})\sum\frac{1}{n_{2j}} & -\frac{1}{rc}\sum\frac{1}{n_{rj}} & -\frac{1}{rc}\sum\frac{1}{n_{i1}} & -\frac{1}{rc}\sum\frac{1}{n_{i2}} & -\frac{1}{rc}\sum\frac{1}{n_{ic}} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{rc}\sum\frac{1}{n_{1j}} & -\frac{1}{rc}\sum\frac{1}{n_{2j}} & \frac{1}{c}(1-\frac{1}{r})\sum\frac{1}{n_{rj}} & -\frac{1}{rc}\sum\frac{1}{n_{i1}} & -\frac{1}{rc}\sum\frac{1}{n_{i2}} & -\frac{1}{rc}\sum\frac{1}{n_{ic}} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{rc}\sum\frac{1}{n_{1j}} & -\frac{1}{rc}\sum\frac{1}{n_{2j}} & -\frac{1}{rc}\sum\frac{1}{n_{rj}} & \frac{1}{r}(1-\frac{1}{c})\sum\frac{1}{n_{i1}} & -\frac{1}{rc}\sum\frac{1}{n_{i2}} & -\frac{1}{rc}\sum\frac{1}{n_{ic}} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{rc}\sum\frac{1}{n_{1j}} & -\frac{1}{rc}\sum\frac{1}{n_{2j}} & -\frac{1}{rc}\sum\frac{1}{n_{rj}} & -\frac{1}{rc}\sum\frac{1}{n_{i1}} & \frac{1}{r}(1-\frac{1}{c})\sum\frac{1}{n_{i2}} & -\frac{1}{rc}\sum\frac{1}{n_{ic}} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{rc}\sum\frac{1}{n_{1j}} & -\frac{1}{rc}\sum\frac{1}{n_{2j}} & -\frac{1}{rc}\sum\frac{1}{n_{rj}} & -\frac{1}{rc}\sum\frac{1}{n_{i1}} & -\frac{1}{rc}\sum\frac{1}{n_{i2}} & \frac{1}{r}(1-\frac{1}{c})\sum\frac{1}{n_{ic}} \\
1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/n_{11} & 0 & 0 & 1/n_{11} & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/n_{12} & 0 & 0 & 0 & 1/n_{12} & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/n_{1c} & 0 & 0 & 0 & 0 & 1/n_{1c} \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/n_{21} & 0 & 1/n_{21} & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/n_{22} & 0 & 0 & 1/n_{22} & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/n_{2c} & 0 & 0 & 0 & 1/n_{2c} \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/n_{r1} & 1/n_{r1} & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1/n_{r2} & 0 & 1/n_{r2} & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1/n_{rc} & 0 & 0 & 1/n_{rc} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1/c & 1/c & 1/c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1/c & 1/c & 1/c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1/c & 1/c & 1/c \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/r & 1/r & 1/r & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/r & 1/r & 1/r & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/r & 1/r & 1/r & 0 & 0 & 1
\end{array}\right)
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\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

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\begin{pmatrix}
\frac{1}{rc}\sum\frac{1}{n_{1j}} & \frac{1}{rc}\sum\frac{1}{n_{2j}} & \frac{1}{rc}\sum\frac{1}{n_{rj}} & \frac{1}{rc}\sum\frac{1}{n_{i1}} & \frac{1}{rc}\sum\frac{1}{n_{i2}} & \frac{1}{rc}\sum\frac{1}{n_{ic}} \\
\frac{1}{c}(1-\frac{1}{r})\sum\frac{1}{n_{1j}} & -\frac{1}{rc}\sum\frac{1}{n_{2j}} & -\frac{1}{rc}\sum\frac{1}{n_{rj}} & -\frac{1}{rc}\sum\frac{1}{n_{i1}} & -\frac{1}{rc}\sum\frac{1}{n_{i2}} & -\frac{1}{rc}\sum\frac{1}{n_{ic}} \\
-\frac{1}{rc}\sum\frac{1}{n_{1j}} & \frac{1}{c}(1-\frac{1}{r})\sum\frac{1}{n_{2j}} & -\frac{1}{rc}\sum\frac{1}{n_{rj}} & -\frac{1}{rc}\sum\frac{1}{n_{i1}} & -\frac{1}{rc}\sum\frac{1}{n_{i2}} & -\frac{1}{rc}\sum\frac{1}{n_{ic}} \\
-\frac{1}{rc}\sum\frac{1}{n_{1j}} & -\frac{1}{rc}\sum\frac{1}{n_{2j}} & \frac{1}{c}(1-\frac{1}{r})\sum\frac{1}{n_{rj}} & -\frac{1}{rc}\sum\frac{1}{n_{i1}} & -\frac{1}{rc}\sum\frac{1}{n_{i2}} & -\frac{1}{rc}\sum\frac{1}{n_{ic}} \\
-\frac{1}{rc}\sum\frac{1}{n_{1j}} & -\frac{1}{rc}\sum\frac{1}{n_{2j}} & -\frac{1}{rc}\sum\frac{1}{n_{rj}} & \frac{1}{r}(1-\frac{1}{c})\sum\frac{1}{n_{i1}} & -\frac{1}{rc}\sum\frac{1}{n_{i2}} & -\frac{1}{rc}\sum\frac{1}{n_{ic}} \\
-\frac{1}{rc}\sum\frac{1}{n_{1j}} & -\frac{1}{rc}\sum\frac{1}{n_{2j}} & -\frac{1}{rc}\sum\frac{1}{n_{rj}} & -\frac{1}{rc}\sum\frac{1}{n_{i1}} & \frac{1}{r}(1-\frac{1}{c})\sum\frac{1}{n_{i2}} & -\frac{1}{rc}\sum\frac{1}{n_{ic}} \\
-\frac{1}{rc}\sum\frac{1}{n_{1j}} & -\frac{1}{rc}\sum\frac{1}{n_{2j}} & -\frac{1}{rc}\sum\frac{1}{n_{rj}} & -\frac{1}{rc}\sum\frac{1}{n_{i1}} & -\frac{1}{rc}\sum\frac{1}{n_{i2}} & \frac{1}{r}(1-\frac{1}{c})\sum\frac{1}{n_{ic}} \\
1/n_{11}-\frac{1}{c}(1-\frac{1}{r})\sum\frac{1}{n_{1j}} & 1/rc\sum 1/n_{2j} & 1/rc\sum 1/n_{rj} & 1/n_{11}-\frac{1}{r}(1-\frac{1}{c})\sum\frac{1}{n_{i1}} & 1/rc\sum 1/n_{i2} & 1/rc\sum 1/n_{ic} \\
1/n_{12}-\frac{1}{c}(1-\frac{1}{r})\sum\frac{1}{n_{1j}} & 1/rc\sum 1/n_{2j} & 1/rc\sum 1/n_{rj} & 1/rc\sum 1/n_{i1} & 1/n_{12}-\frac{1}{r}(1-\frac{1}{c})\sum\frac{1}{n_{i2}} & 1/rc\sum 1/n_{ic} \\
1/n_{1c}-\frac{1}{c}(1-\frac{1}{r})\sum\frac{1}{n_{1j}} & 1/rc\sum 1/n_{2j} & 1/rc\sum 1/n_{rj} & 1/rc\sum 1/n_{i1} & 1/rc\sum 1/n_{i2} & 1/n_{1c}-\frac{1}{r}(1-\frac{1}{c})\sum\frac{1}{n_{ic}} \\
1/rc\sum 1/n_{1j} & 1/n_{21}-\frac{1}{c}(1-\frac{1}{r})\sum\frac{1}{n_{2j}} & 1/rc\sum 1/n_{rj} & 1/n_{21}-\frac{1}{r}(1-\frac{1}{c})\sum\frac{1}{n_{i1}} & 1/rc\sum 1/n_{i2} & 1/rc\sum 1/n_{ic} \\
1/rc\sum 1/n_{1j} & 1/n_{22}-\frac{1}{c}(1-\frac{1}{r})\sum\frac{1}{n_{2j}} & 1/rc\sum 1/n_{rj} & 1/rc\sum 1/n_{i1} & 1/n_{22}-\frac{1}{r}(1-\frac{1}{c})\sum\frac{1}{n_{i2}} & 1/rc\sum 1/n_{ic} \\
1/rc\sum 1/n_{1j} & 1/n_{2c}-\frac{1}{c}(1-\frac{1}{r})\sum\frac{1}{n_{2j}} & 1/rc\sum 1/n_{rj} & 1/rc\sum 1/n_{i1} & 1/rc\sum 1/n_{i2} & 1/n_{2c}-\frac{1}{r}(1-\frac{1}{c})\sum\frac{1}{n_{ic}} \\
1/rc\sum 1/n_{1j} & 1/rc\sum 1/n_{2j} & 1/n_{r1}-\frac{1}{c}(1-\frac{1}{r})\sum\frac{1}{n_{rj}} & 1/n_{r1}-\frac{1}{r}(1-\frac{1}{c})\sum\frac{1}{n_{i1}} & 1/rc\sum 1/n_{i2} & 1/rc\sum 1/n_{ic} \\
1/rc\sum 1/n_{1j} & 1/rc\sum 1/n_{2j} & 1/n_{r2}-\frac{1}{c}(1-\frac{1}{r})\sum\frac{1}{n_{rj}} & 1/rc\sum 1/n_{i1} & 1/n_{r2}-\frac{1}{r}(1-\frac{1}{c})\sum\frac{1}{n_{i2}} & 1/rc\sum 1/n_{ic} \\
1/rc\sum 1/n_{1j} & 1/rc\sum 1/n_{2j} & 1/n_{rc}-\frac{1}{c}(1-\frac{1}{r})\sum\frac{1}{n_{rj}} & 1/rc\sum 1/n_{i1} & 1/rc\sum 1/n_{i2} & 1/n_{rc}-\frac{1}{r}(1-\frac{1}{c})\sum\frac{1}{n_{ic}} \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1/c & 1/c & 1/c \\
0 & 1 & 0 & 1/c & 1/c & 1/c \\
0 & 0 & 1 & 1/c & 1/c & 1/c \\
1/r & 1/r & 1/r & 1 & 0 & 0 \\
1/r & 1/r & 1/r & 0 & 1 & 0 \\
1/r & 1/r & 1/r & 0 & 0 & 1
\end{pmatrix}
$$

By row operations, we now have taken the augmented X'X matrix to the form:

$$\begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{X'X} & \mathbf{H'} \\ \mathbf{H} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{A}_{11}\mathbf{H'} \\ \mathbf{0} & \mathbf{A}_{21}\mathbf{H'} \end{pmatrix}$$

We still need:

$$\begin{pmatrix} \mathbf{B}_{11} & \mathbf{B}_{21}' \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{X'X} & \mathbf{H'} \\ \mathbf{H} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

but:

$\mathbf{B}_{11} = \mathbf{A}_{11} - \mathbf{A}_{11}\mathbf{H'}(\mathbf{A}_{21}\mathbf{H'})^{-1}\mathbf{A}_{21}$ and $\mathbf{B}_{21} = (\mathbf{A}_{21}\mathbf{H'})^{-1}\mathbf{A}_{21}$

$\mathbf{A}_{21}\mathbf{H'}$ is an (r+c+1)x(r+c+1) matrix of the form $\begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{pmatrix}$ where the lower right hand matrix is of the form:

$$\begin{pmatrix} 1 & 0 & 0 & 1/c & 1/c & 1/c \\ 0 & 1 & 0 & 1/c & 1/c & 1/c \\ 0 & 0 & 1 & 1/c & 1/c & 1/c \\ 1/r & 1/r & 1/r & 1 & 0 & 0 \\ 1/r & 1/r & 1/r & 0 & 1 & 0 \\ 1/r & 1/r & 1/r & 0 & 0 & 1 \end{pmatrix}$$

The inverse is:

$$\begin{pmatrix} 2 & 1 & 1 & -r/c & -r/c & -r/c \\ 1 & 2 & 1 & -r/c & -r/c & -r/c \\ 1 & 1 & 2 & -r/c & -r/c & -r/c \\ -1 & -1 & -1 & 1+(r-1)/c & (r-1)/c & (r-1)/c \\ -1 & -1 & -1 & (r-1)/c & 1+(r-1)/c & (r-1)/c \\ -1 & -1 & -1 & (r-1)/c & (r-1)/c & 1+(r-1)/c \end{pmatrix}$$

an (r+c-1)x((r+c-1) matrix: upper left matrix is (r-1)x(r-1); lower right matrix is cxc

$$
\begin{pmatrix}
N & n_{1.} & n_{2.} & n_{r.} & n_{.1} & n_{.2} & n_{.c} & 0 & 0 \\
n_{1.} & n_{1.} & 0 & 0 & n_{11} & n_{12} & n_{1c} & 1 & 0 \\
n_{2.} & 0 & n_{2.} & 0 & n_{21} & n_{22} & n_{2c} & 1 & 0 \\
n_{r.} & 0 & 0 & n_{r.} & n_{r1} & n_{r2} & n_{rc} & 1 & 0 \\
n_{.1} & n_{11} & n_{21} & n_{r1} & n_{.1} & 0 & 0 & 0 & 1 \\
n_{.2} & n_{12} & n_{12} & n_{r2} & 0 & n_{.2} & 0 & 0 & 1 \\
n_{.c} & n_{1c} & n_{2c} & n_{rc} & 0 & 0 & n_{.c} & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix} m \\ r_1 \\ r_2 \\ r_r \\ c_1 \\ c_2 \\ c_c \\ \lambda_r \\ \lambda_c \end{pmatrix}
=
\begin{pmatrix} T \\ T_{1.} \\ T_{2.} \\ T_{r.} \\ T_{.1} \\ T_{.2} \\ T_{.c} \\ 0 \\ 0 \end{pmatrix}
$$

$$
\begin{pmatrix}
1/N & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-n_{1.}/N & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-n_{2.}/N & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-n_{r.}/N & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-n_{.1}/N & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-n_{.2}/N & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-n_{.c}/N & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
N & n_{1.} & n_{2.} & n_{r.} & n_{.1} & n_{.2} & n_{.c} & 0 & 0 \\
n_{1.} & n_{1.} & 0 & 0 & n_{11} & n_{12} & n_{1c} & 1 & 0 \\
n_{2.} & 0 & n_{2.} & 0 & n_{21} & n_{22} & n_{2c} & 1 & 0 \\
n_{r.} & 0 & 0 & n_{r.} & n_{r1} & n_{r2} & n_{rc} & 1 & 0 \\
n_{.1} & n_{11} & n_{21} & n_{r1} & n_{.1} & 0 & 0 & 0 & 1 \\
n_{.2} & n_{12} & n_{12} & n_{r2} & 0 & n_{.2} & 0 & 0 & 1 \\
n_{.c} & n_{1c} & n_{2c} & n_{rc} & 0 & 0 & n_{.c} & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & n_{1.}/N & n_{2.}/N & n_{r.}/N & n_{.1}/N & n_{.2}/N & n_{.c}/N & 0 & 0 \\
0 & n_1 - n_{1.}^2/N & -n_{1.}n_{2.}/N & -n_{1.}n_{r.}/N & n_{11}-n_{1.}n_{.1}/N & n_{12}-n_{1.}n_{.2}/N & n_{1c}-n_{1.}n_{.c}/N & 1 & 0 \\
0 & -n_{1.}n_{2.}/N & n_{2.}-n_{2.}^2/N & -n_{2.}n_{r.}/N & n_{21}-n_{2.}n_{.1}/N & n_{22}-n_{2.}n_{.2}/N & n_{2c}-n_{2.}n_{.c}/N & 1 & 0 \\
0 & -n_{1.}n_{r.}/N & -n_{2.}n_{r.}/N & n_{r.}-n_{r.}^2/N & n_{r1}-n_{r.}n_{.1}/N & n_{r2}-n_{r.}n_{.2}/N & n_{rc}-n_{r.}n_{.c}/N & 1 & 0 \\
0 & n_{11}-n_{1.}n_{.1}/N & n_{21}-n_{2.}n_{.1}/N & n_{r1}-n_{r.}n_{.1}/N & n_{.1}-n_{.1}^2/N & -n_{.1}n_{.2}/N & -n_{.1}n_{.c}/N & 0 & 1 \\
0 & n_{12}-n_{1.}n_{.2}/N & n_{22}-n_{2.}n_{.2}/N & n_{r2}-n_{r.}n_{.2}/N & -n_{.1}n_{.2}/N & n_{.2}-n_{.2}^2/N & -n_{.2}n_{.c}/N & 0 & 1 \\
0 & n_{1c}-n_{1.}n_{.c}/N & n_{2c}-n_{2.}n_{.c}/N & n_{rc}-n_{r.}n_{.c}/N & -n_{.1}n_{.c}/N & -n_{.2}n_{.c}/N & n_{.c}-n_{.c}^2/N & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}
$$

$$\sum_{k=1}^{r} r_k (T_{k.} - \frac{n_{k.}}{N} T) + \sum_{l=1}^{c} c_{.l} (T_{.l} - \frac{n_{.l}}{N} T) = \sum_{k=1}^{r} r_k T_{k.} + \sum_{l=1}^{c} c_{.l} T_{.l} - \frac{T}{N} (\sum_{k=1}^{r} r_k n_{k.} + \sum_{l=1}^{c} c_{.l} n_{.l})$$

$$= \sum_{k=1}^{r} r_k T_{k.} + \sum_{l=1}^{c} c_{.l} T_{.l} - \frac{T}{N} (T - Nm)$$

$$= mT + \sum_{k=1}^{r} r_k T_{k.} + \sum_{l=1}^{c} c_{.l} T_{.l} - \frac{T^2}{N}$$

<u>Proportional Frequencies</u>

$$\begin{pmatrix}
1 & \frac{\lambda_1}{\Lambda} & \frac{\lambda_2}{\Lambda} & \frac{\lambda_r}{\Lambda} & \frac{\mu_1}{M} & \frac{\mu_2}{M} & \frac{\mu_c}{M} & 0 & 0 \\
0 & N\frac{\lambda_1}{\Lambda}(1-\frac{\lambda_1}{\Lambda}) & -N\frac{\lambda_1}{\Lambda}\frac{\lambda_2}{\Lambda} & -N\frac{\lambda_1}{\Lambda}\frac{\lambda_r}{\Lambda} & 0 & 0 & 0 & 1 & 0 \\
0 & -N\frac{\lambda_1}{\Lambda}\frac{\lambda_2}{\Lambda} & N\frac{\lambda_2}{\Lambda}(1-\frac{\lambda_2}{\Lambda}) & -N\frac{\lambda_2}{\Lambda}\frac{\lambda_r}{\Lambda} & 0 & 0 & 0 & 1 & 0 \\
0 & -N\frac{\lambda_1}{\Lambda}\frac{\lambda_r}{\Lambda} & -N\frac{\lambda_2}{\Lambda}\frac{\lambda_r}{\Lambda} & N\frac{\lambda_r}{\Lambda}(1-\frac{\lambda_r}{\Lambda}) & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & N\frac{\mu_1}{M}(1-\frac{\mu_1}{M}) & -N\frac{\mu_1}{M}\frac{\mu_2}{M} & -N\frac{\mu_1}{M}\frac{\mu_c}{M} & 0 & 1 \\
0 & 0 & 0 & 0 & -N\frac{\mu_1}{M}\frac{\mu_2}{M} & N\frac{\mu_2}{M}(1-\frac{\mu_2}{M}) & -N\frac{\mu_2}{M}\frac{\mu_c}{M} & 0 & 1 \\
0 & 0 & 0 & 0 & -N\frac{\mu_1}{M}\frac{\mu_c}{M} & -N\frac{\mu_2}{M}\frac{\mu_c}{M} & N\frac{\mu_c}{M}(1-\frac{\mu_c}{M}) & 0 & 1 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\Lambda}{\lambda_1 N} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\Lambda}{\lambda_2 N} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\Lambda}{\lambda_r N} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{M}{\mu_1 N} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{M}{\mu_2 N} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{M}{\mu_c N} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$
\begin{pmatrix}
1 & \frac{\lambda_1}{\Lambda} & \frac{\lambda_2}{\Lambda} & \frac{\lambda_r}{\Lambda} & \frac{\mu_1}{M} & \frac{\mu_2}{M} & \frac{\mu_c}{M} & 0 & 0 \\
0 & 1-\frac{\lambda_1}{\Lambda} & -\frac{\lambda_2}{\Lambda} & -\frac{\lambda_r}{\Lambda} & 0 & 0 & 0 & \frac{\Lambda}{\lambda_1 N} & 0 \\
0 & -\frac{\lambda_1}{\Lambda} & 1-\frac{\lambda_2}{\Lambda} & -\frac{\lambda_r}{\Lambda} & 0 & 0 & 0 & \frac{\Lambda}{\lambda_2 N} & 0 \\
0 & -\frac{\lambda_1}{\Lambda} & -\frac{\lambda_2}{\Lambda} & 1-\frac{\lambda_r}{\Lambda} & 0 & 0 & 0 & \frac{\Lambda}{\lambda_r N} & 0 \\
0 & 0 & 0 & 0 & 1-\frac{\mu_1}{M} & -\frac{\mu_2}{M} & -\frac{\mu_c}{M} & 0 & \frac{M}{\mu_1 N} \\
0 & 0 & 0 & 0 & -\frac{\mu_1}{M} & 1-\frac{\mu_2}{M} & -\frac{\mu_c}{M} & 0 & \frac{M}{\mu_2 N} \\
0 & 0 & 0 & 0 & -\frac{\mu_1}{M} & -\frac{\mu_2}{M} & 1-\frac{\mu_c}{M} & 0 & \frac{M}{\mu_c N} \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}
$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\frac{1}{r} & -\frac{1}{r} & -\frac{1}{r} & 0 & 0 & 0 & \frac{1}{r} & 0 \\ 0 & -\frac{1}{r} & 1-\frac{1}{r} & -\frac{1}{r} & 0 & 0 & 0 & \frac{1}{r} & 0 \\ 0 & -\frac{1}{r} & -\frac{1}{r} & 1-\frac{1}{r} & 0 & 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & 0 & 1-\frac{1}{c} & -\frac{1}{c} & -\frac{1}{c} & 0 & \frac{1}{c} \\ 0 & 0 & 0 & 0 & -\frac{1}{c} & 1-\frac{1}{c} & -\frac{1}{c} & 0 & \frac{1}{c} \\ 0 & 0 & 0 & 0 & -\frac{1}{c} & -\frac{1}{c} & 1-\frac{1}{c} & 0 & \frac{1}{c} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$
\begin{pmatrix}
1 & \frac{\lambda_1}{\Lambda} & \frac{\lambda_2}{\Lambda} & \frac{\lambda_r}{\Lambda} & \frac{\mu_1}{M} & \frac{\mu_2}{M} & \frac{\mu_c}{M} & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{\Lambda}{N}(\frac{1}{\lambda_1}-\frac{1}{r}\sum\frac{1}{\lambda_k}) & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{\Lambda}{N}(\frac{1}{\lambda_2}-\frac{1}{r}\sum\frac{1}{\lambda_k}) & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{\Lambda}{N}(\frac{1}{\lambda_1}-\frac{1}{r}\sum\frac{1}{\lambda_k}) & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{M}{N}(\frac{1}{\mu_1}-\frac{1}{c}\sum\frac{1}{\mu_j}) \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{M}{N}(\frac{1}{\mu_2}-\frac{1}{c}\sum\frac{1}{\mu_j}) \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{M}{N}(\frac{1}{\mu_c}-\frac{1}{c}\sum\frac{1}{\mu_j}) \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}
$$