THE FAMILY OF MODULATED BETA DISTRIBUTIONS

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Summary

Through the use of a monotonic transformation, we present a new family of distributions defined on the unit interval that is derived from the family of beta distributions. This family is being called the modulated beta family since the members of the family are also obtainable through an application of likelihood modulation. Various applications and illustrations are included here along with several natural generalizations.

2. The Modulated Beta Distributions

The family of beta distributions defined on the interval $(0,1)$ have probability density functions

$$f_\beta(t;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} t^{\alpha-1}(1-t)^{\beta-1}, \quad 0<t<1 \tag{2.1}$$

with parameters $\alpha>0$ and $\beta>0$. By invoking the monotone transformation $X = T/[1-\kappa(1-T)]$ for $\kappa\epsilon[0,1)$ where T has the beta (α,β) distribution, one obtains a family of distributions on $(0,1)$ which will be called the family of modulated beta distributions. The probability density function for X is given by

$$f_{m\beta}(x;\alpha,\beta,\kappa) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}(1-\kappa)^\alpha \frac{x^{\alpha-1}(1-x)^{\beta-1}}{(1-\kappa x)^{\alpha+\beta}}, \quad 0<x<1. \tag{2.2}$$

As before $\alpha, \beta>0$; the parameter $\kappa\epsilon[0,1)$ will be referred to as the modulation parameter. When $\kappa = 0$ the modulated beta (α,β,κ) distribution reduces to the beta (α,β) distribution. The transformation from T to X is monotone and defined on $(0,1)$ for all $\kappa<1$. However, when $\kappa<0$ one can replace X by $(1-X)$ to obtain a random variable having the modulated beta (β,α,κ^*) distribution where $\kappa^* = \kappa/(\kappa-1)\epsilon(0,1)$ so that attention can be restricted to the case $\kappa\epsilon[0,1)$ without losing generality.

The family of modulated beta distributions with fixed α and β is stochastically ordered by the value of the modulation parameter κ. Specifically, if X and X^* are distributed modulated beta with parameters κ and κ^* with $\kappa>\kappa^*$, then X is stochastically larger than X^*. As κ increases on $[0,1)$ the distribution of X becomes increasingly concentrated at the upper end of the range $(0,1)$.

An important feature of the family of modulated beta distributions is the fact that it is closed under the monotone transformation $X = X^*/(1-h(1-X^*))$ or its inverse $X^* = (1-X)h/(1-Xh)$ for $h\epsilon[0,1)$. If either X or X^* is modulated beta then so is the other; the modulation parameters κ and κ^* satisfy the relationship $(1-\kappa) = (1-\kappa^*)(1-h)$ where all of $\kappa_X\kappa^*$ and $h\epsilon[0,1)$. Similarly, for any two members of the family of modulated beta distributions there exists an $h\epsilon[0,1)$ such that the transformation transforms one member into the other. In particular the transformation used to define the family of modulated beta distributions corresponds to $\kappa^* = 0$ and $h = \kappa$.

Associated with the family of beta distributions is the family of inverted beta distributions, also known as the beta-2 distributions. If $T \sim Beta(\alpha,\beta)$ then $U = T/(1-T)$ has the inverted beta (α,β) distribution with probability density function

$$f_{i\beta}(u;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}, \quad u>0 \tag{2.3}$$

Applying the same transformation to $X \sim$ Modulated Beta (α,β,κ), namely $Y = X/(1-X)$, one obtains a modulated inverted beta distribution with probability density function

$$f_{mi\beta}(y;\alpha,\beta,\kappa) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}(1-\kappa)^{\alpha}\frac{y^{\alpha-1}}{(1+(1-\kappa)y)^{\alpha+\beta}}, \quad y>0. \tag{2.4}$$

Note, however, that $Y = X/(1-X) = (1-\kappa)^{-1}T/(1-T)$ so that the distribution of Y constitutes a rescaling of the inverted beta (α,β) distribution with scale factor $1/(1-\kappa)$.

The modulated beta (α,β,κ) distribution can be represented as a mixture of beta distributions where the mixing distribution is the negative binomial (α,κ) distribution with probability mass function

$$p_{nb}(j;\alpha,\kappa) = \frac{\Gamma(\alpha+j)}{\Gamma(\alpha)j!}\kappa^j(1-\kappa)^\alpha, \quad j = 0,1,2,\ldots\ .$$

This follows from the series expansion of $(1-t)^{-\nu}$ valid for $|t|<1$ when $\nu>0$. The probability density of the modulated beta (α,β,κ) distribution then has the form

$$f_{m\beta}(x;\alpha,\beta,\kappa) = \sum_{j=0}^{\infty} p_{nb}(j;\alpha,\kappa) f_\beta(x;\alpha t j,\beta)$$

Using the same argument, the density of the modulated (scaled) inverted beta (α,β,κ) distribution can be represented as

$$f_{mi\beta}(y;\alpha,\beta,\kappa) = \sum_{j=0}^{\infty} p_{nb}(j;\alpha,\kappa) f_{i\beta}(y;\alpha t j,\beta).$$

The representation of the modulated beta distribution as a negative binomial mixture of beta distributions provides a connection between the modulated beta distribution and the noncentral beta distribution. The latter distribution can be represented as a mixture of beta distributions where the mixing distribution is the Poisson (λ) distribution, namely

$$f_{\beta'}(x;\alpha,\beta,\lambda) = \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} f_\beta(x;\alpha+j,\beta).$$

If one assumes that λ has a gamma (α_0,τ_0) distribution, a member of the conjugate family of prior distributions for λ, then the marginal distribution of the mixing variable is the negative binomial (α_0,κ) distribution with $\kappa + 1/(1+\tau_0)$, and hence the marginal distribution of X is an arbitrary negative binomial mixture of beta distributions. In the special case when the prior parameter $\tau_0>0$ is arbitrary but $\alpha_0 = \alpha$, the marginal distribution of X reduces to the modulated beta (α,β,κ) distribution.

The moments of the beta (α,β) distribution are given by

$$E(T^r;\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}\frac{\Gamma(\alpha+r)}{\Gamma(\alpha+\beta+r)} = \frac{(\alpha)_r}{(\alpha+\beta)_r}, \quad r = 1,2,\ldots \tag{2.5}$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is commonly referred to as "a ascending factorial n". When n is a nonnegative integer then

$$(a)_0 = 1 \text{ and } (a)_n = a(a+1)\cdots(a+n-1), \quad n = 1,2,\ldots \; .$$

The moments of the modulated beta (α,β,κ) distribution cannot in general be expressed in closed form. However, using the representation ( ) it follows that:

$$E(X^r;\alpha,\beta,\kappa) = \sum_{j=0}^{\infty} p_{nb}(j;\alpha,\kappa)E(T^r;\alpha+j,\beta)$$

$$= \frac{(\alpha)_r}{(\alpha+\beta)_r}(1-\kappa)^{\alpha}\left\{\sum_{j=0}^{\infty}\frac{(\alpha+\beta)_j(\alpha+r)_j}{(\alpha+\beta+r)_j}\frac{\kappa^j}{j!}\right\}$$

$$= \frac{(\alpha)_r}{(\alpha+\beta)_r}(1-\kappa)^{\alpha}F(\alpha+\beta,\alpha+r;\alpha+\beta+r;\kappa) \tag{2.6}$$

where F denotes the hypergeometric function in κ. The hypergeometric series in Z with parameters a, b and c is defined by

$$F(a,b;c;Z) = \sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j}\frac{Z^j}{j!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{j=0}^{\infty}\frac{\Gamma(a+j)\Gamma(b+j)}{\Gamma(c+j)}\frac{Z^j}{j} \tag{2.7}$$

A comprehensive treatment of the hypergeometric function and its properties may be found in Erdelyi et al. (1981), Volume 1, Chapter 2. In the present context a, b and $c>0$ so that $F(a,b;c;Z)$ is absolutely convergent for $|Z|<1$. Moreover $c>b>0$ so that $F(a,b;c;Z)$ has the integral representation

$$F(a,b;c;Z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{x^{b-1}(1-x)^{c-b-1}}{(1-Zx)^a}dx.$$

An important special case of the hypergeometric function arises when $c = a$ (or equivalently b) in which case

$$F(a,b;a;Z) = \sum_{j=0}^{\infty}\frac{\Gamma(b+j)}{\Gamma(b)}\frac{Z^j}{j!} = (1-Z)^{-b}.$$

Using this special case we can represent the r^{th} moment of the modulated beta (α,β,κ) distribution by

$$E(X^r;\alpha,\beta,\kappa) = \frac{(\alpha)_r}{(\alpha+\beta)_r}\frac{F(\alpha+\beta,\alpha+r;\alpha+\beta+r;\kappa)}{F(\alpha+\beta,\alpha;\alpha+\beta;\kappa)}. \tag{2.8}$$

3. The Likelihood Modulation Method

For certain problems in parametric inference, one can combine knowledge of the distribution of the appropriate statistic at a specific point in the parameter space with a specification of the likelihood based on that statistic in order to determine the distribution of the statistic for all points in the parameter space. This procedure was used in the construction of tests of significance on the circle and the sphere by Watson and Williams (1956) and appears to be implicit in the analysis of dispersion on a sphere given by Fisher (1953). The process of adjusting a specific member of a family of distributions by the likelihood was named **likelihood modulation** by Fraser (1968) who explicitly used the method to obtain the distribution of the correlation coefficient when sampling from a bivariate normal distribution. The first formulation of likelihood modulation as a general method appears in Fraser (1979).

The method of likelihood modulation will now be described in detail. Let $f(\mathbf{x};\theta)$ be a family of joint density (probability) functions for $\mathbf{x} = (x_1, \ldots, x_n)$ indexed by parameter $\theta \epsilon \Theta$, a general parameter space, for which a statistic $t = t(\mathbf{x})$ is known to be sufficient. The likelihood $L(\mathbf{x};\theta)$ can be reexpressed as $L(t;\theta)$ in order to reflect the fact that the likelihood is the same for all $\mathbf{x}$ that yield a particular value t of the sufficient statistic. Indeed, the class of possible likelihood functions can be partitioned into equivalence classes of similarly shaped functions of θ that are in one to one correspondence with the equivalence classes of sample points determined by the minimal sufficient statistic, a property that led Fraser (1976) to refer to the latter as the likelihood statistic. Since

$$L(t;\theta) = h(t)g(t;\theta)$$

where $g(t;\theta)$ is the density (probability) function of the sufficient statistic and $h(t)$ is a constant of proportionality for the likelihood viewed as a function of θ, it follows that for any specified θ_0

$$g(t;\theta) = g(t;\theta_0)L^*(t;\theta)$$

where $L^*(t;\theta)$ is the representative likelihood obtained by standardizing the class of similarly shaped functions relative to θ_0 so that $L^*(t;\theta_0) = 1$ for all t. Hence, if one knows the distribution of a sufficient statistic $t(\mathbf{x})$ at a specified θ_0 and one knows the representative likelihood $L^*(t;\theta)$, then modulation of the specified distribution by the representative likelihood yields the distribution of the statistic $t(\mathbf{x})$ for all $\theta \epsilon \Theta$.

The family of distributions that was introduced in Section 2 by transformation of a random variable having the beta (α,β) distribution can also be obtained through the use of likelihood modulation. Assume that a family of joint probability distributions admits a statistic X that is sufficient for the parameter $\kappa\epsilon[0,1)$, and that the distribution of X when $\kappa=0$ is known to be a beta (α,β) distribution with (α,β) specified. In addition, assume that the representative likelihood has the form

$$L^*(x;\kappa) = (1-\kappa)^{\alpha}(1-\kappa x)^{-(\alpha+\beta)}$$

where $L^*(x;0)=1$ for all $x\epsilon(0,1)$. Then modulation of the density when $\kappa=0$ by the representative likelihood yields the family of densities (2.2) for the modulated beta distribution. This family has been obtained by Fick and Davidson (1986) in the investigation of the residuals from linear regression under a variance inflation model.

4. Further Families of Generalized Beta Distributions

The family of modulated beta distributions introduced in Section 2 was observed to be closed under the monotone transformation $X = X^*/(1-h(1-X^*))$ for $h\epsilon[0,1)$. Although the density function of the modulated beta distribution has a closed form, it can also be represented as a negative binomial mixture of beta densities. This section examines ways in which the family of modulated beta distributions can be generalized. Two distinct generalized families are suggested that lead to a further generalization that includes both families.

The first generalized family corresponds to extending the closed form (2.2) for the density of the modulated beta distribution to obtain the following density

$$f_1(x;\alpha,\beta,\gamma,\kappa) = \frac{1}{F(\gamma,\alpha;\alpha+\beta;\kappa)}\,\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,\frac{x^{\alpha-1}(1-x)^{\beta-1}}{(1-\kappa x)^{\gamma}},\quad 0<x<1 \qquad (4.1)$$

where α, β, $\gamma>0$ and $\kappa\epsilon[0,1)$, and where F denotes the hypergeometric function in variable κ with parameters γ, α and $\alpha+\beta$. By use of the series expansion of $(1-\kappa x)^{-\gamma}$, the density (4.1) can be represented as a mixture of beta distributions. In this case the mixing distribution is a generalized hypergeometric distribution in the class given by Kemp (1968) and catalogued by Dacey (1972) and has probability function

$$p_{hyp}(j;a,b,c,\kappa) = \frac{1}{F(a,b;c;\kappa)} \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+j)\Gamma(b+j)}{\Gamma(c+j)} \frac{\kappa^j}{j!}$$

$$= \frac{1}{F(a,b;c;\kappa)} \frac{(a)_j(b)_j}{(c)_j} \frac{\kappa^j}{j!}, \quad j = 0,1,2,\ldots$$

where a, b and $c>0$ and $\kappa\epsilon[0,1)$. The probability generating function for the generalized hypergeometric distribution is

$$G(t) = F(a,b;c;\kappa t)/F(a,b;c;\kappa).$$

The density (4.1) can then be written

$$f_1(x;\alpha,\beta,\gamma,\kappa) = \sum_{j=0}^{\infty} p_{hyp}(j;\gamma,\alpha,\alpha+\beta;\kappa) f_\beta(x;\alpha+j,\beta), \quad 0<x<1.$$

One obtains the modulated beta distribution when $\gamma = \alpha+\beta$ in which case the generalized hypergeometric mixing distribution reduces to the negative binomial (α,κ) distribution.

The second generalized family corresponds to arbitrary negative binomial mixtures of beta distributions with densities given by

$$f_2(x;\alpha,\beta,\nu,\kappa) = \sum_{j=0}^{\infty} p_{nb}(j;\nu,\kappa) f_\beta(x;\alpha+j,\beta), \quad 0<x<1 \tag{4.2}$$

for $\nu>0$, and where as before α, $\beta>0$ and $\kappa\epsilon[0,1)$. The density (4.2) can be expressed in closed form if and only if $\nu = \alpha$ from which one obtains the modulated beta distribution, or when $\kappa = 0$ which yields the beta distribution. The density (4.2) can be represented as follows:

$$f_2(x;\alpha,\beta,\nu,\kappa) = (1-\kappa)^{\nu}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}F(\alpha+\beta,\nu;\alpha;\kappa x), \quad 0<x<1$$

$$= (1-\kappa)^{\nu}f_{\beta}(x;\alpha,\beta)F(\alpha+\beta,\nu;\alpha;\kappa x)$$

where F denotes the hypergeometric function in variable κx with parameters $\alpha+\beta$, ν and α.

The family of modulated beta distributions constitutes the largest subfamily contained in both the generalized families of densities (4.1) and (4.2). A further generalization to include both these families of distributions can now be proposed. This family has densities

$$g(x;\alpha,\beta,\gamma,\nu,\kappa) = \sum_{j=0}^{\infty}p_{hyp}(j;\gamma,\nu,\alpha+\beta,\kappa)f_{\beta}(x;\alpha+j,\beta)$$

$$= \frac{F(\gamma,\nu;\alpha;\kappa x)}{F(\gamma,\nu;\alpha+\beta,\kappa)}f_{\beta}(x;\alpha,\beta) \tag{4.3}$$

with α, β, γ and $\nu>0$ and $\kappa\epsilon[0,1)$. Note that the hypergeometric function in the numerator involves the random variable x as well as the modulation parameter κ, while that in the denominator depends only on κ. From this most general family of densities (4.3), one obtains the subfamily (4.1) when $\gamma = \alpha+\beta$, the second subfamily (4.2) when $\nu = 2$, and the modulated beta family of distributions (2.2) when both $\gamma = \alpha+\beta$ and $\nu = \alpha$.